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Third-Order Solution to the Main Problem in Satellite Theory

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In the present paper we announce a completely analytic closed-form third-order solution to the main problem in the theory of an artificial satellite. This is the first time an analytic solution of the main problem has been produced to order 3 which is valid for satellites with any eccentricity $0 \le e < 1$. The solution is accomplished by constructing a progression of three canonical transformations from the state variables to a set of action-angle variables in which the Hamiltonian for the problem is a function of the action variables only. The transformed Hamiltonian is developed, without omitting terms, to order 4 in the small parameter $\epsilon = -J_2$; by way of verification it was found to agree through order 3 with the theory of Brouwer as extended by Kozai. The algebraic expressions for these transformations were produced by computer in a very compact form; conciseness is achieved in two ways, by eliminating the parallax and by controlling the computer automated calculations so as to avoid infinite series expansions in the eccentricity. Our programs prove that the main problem in satellite theory can be solved in closed form to order 3.

I. Introduction

IN the present paper we develop a completely analytic third-order solution to the main problem in the theory of an artificial satellite. The theories of Deprit and Rom[1] and Claes[2] rely on eccentricity expansions of the perturbation to effect the integrations necessary to eliminate the short period terms. A more recent report by Kinoshita[3] expands upon the theory of Deprit and Rom by including the J_3 and J_4 zonal harmonics. But there again the perturbations were expanded as truncated series in the eccentricity, limiting that theory to near circular orbits.

The secular and the long period Hamiltonians have been produced to order 4 in the small parameter $\epsilon = -J_2$. Here we denote the long period Hamiltonian as that derived from the main problem after averaging over the mean anomaly. By maintaining Delaunay's elements as the phase space coordinates, we were able to compare directly the first three orders of our averaged Hamiltonians with the earlier work of Brouwer,[4] and its extension by Kozai.[5] Comparison with Kozai through order 3 established that our long period Hamiltonians are equivalent and our secular Hamiltonians are identical (save for what we assume to be a misprint in Kozai's formula 6.21 where p should be replaced by P). On the other hand, comparison formula by formula with Kinoshita is not possible since the latter produces only expressions truncated in the eccentricity.

Our theory has been kept nonsingular for all eccentricities and all inclinations except for the so-called critical inclination. Of course the singularities of small inclination and small eccentricity will occur if the user insists on expressing the state of the system by way of Delaunay's elements rather than a set of uniformly regular variables such as those defined by Deprit and Rom[1] or Hoots.[6] Our reduced Hamiltonian establishes that to order 4 small eccentricities do not constitute intrinsic singularities and likewise it also shows that at least to order 3 small inclinations are not intrinsic singularities.

At last Brouwer's dream of a solution to the main problem developed automatically by machine has been made real. Several authors (e.g., Aksnes[7] and Jefferys[8]) have considered that sort of automation. Here, however, the task has been brought to completion. We use a revised version of MAO (Mechanized Algebraic Operations)[9] written in PL/I by the second author and the Fortran package for algebraic manipulation developed by R. Dansenbrock.[10]

The elimination of the short period terms was accomplished by constructing two canonical transformations of the Lie type rather than a unique von Zeipel tranformation as done by Brouwer and Kozai. However the total number of terms in the generators for our short period transformations contain only 25% of the number of terms to be found in the corresponding single transformation. In fact, so concise were the expressions for our generators that at order 4 we went beyond averaging over the mean anomaly to construct the corresponding terms in each of the generators. Indeed, armed with these generators one could easily, in principle at least but discounting the computational complexity, compute the fifth order in the long period Hamiltonian and thus construct a fourth-order theory for the main problem. We note that at third order our generators contain 112 and 125 terms, respectively, as compared to the more than 2000 terms required for the third order in Kozai's von Zeipel generator. This will drastically reduce the computational times required for orbit prediction, which is especially important in the area of real time processing. The increase in efficiency will become even more evident when the same technique is applied to the Hamiltonian consisting of the main problem plus higher-order zonal harmonics.

In the last section of the paper we complete the third-order theory by constructing yet another canonical transformation of the Lie type to eliminate the long period terms and thus arrive at the secular Hamiltonian.

II. Short Period Elimination

The Hamiltonian for the main problem in the theory of an artificial satellite is given in Whittaker's variables by

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$$
\begin{aligned}
&M(r,\theta,-,R,\Theta,N)=M_0+\epsilon M_1\\
&M_0=\tfrac{1}{2}(R^2+\Theta^2/r^2)-\mu/r\\
&M_1=(\mu/r)(\alpha/r)^2(\tfrac{1}{2}-\tfrac{3}{4}s^2+\tfrac{3}{4}s^2\cos 2\theta)\\
&s=\sin I=\sqrt{1-N^2/\Theta^2}
\end{aligned}
\qquad (1)
$$

This defines a Hamiltonian with two degrees of freedom. The right ascension of the node, ν, is ignorable, and $\theta=f+g$ is the argument of latitude. The variable r is the radial distance from the Earth's center of mass to the satellite. The variables R,Θ, and N are the momenta for r,θ, and ν, respectively. The constants μ and α stand for the Keplerian constant and the equatorial radius of the Earth, while I is the inclination of the satellite's orbital plane.

The elimination of the short period terms from Eq. (1) is achieved through the construction of two canonical transformations of the Lie type.

The first transformation is called the elimination of the parallax and has been reported on elsewhere.[11] Briefly we recount that this transformation simplifies Eq. (1) by converting the factors $(\mu/r)(\alpha/r)^2$ to $(\mu/p)(\alpha/r)^2$ while eliminating the true anomaly from the angle θ. The resulting Hamiltonian has been constructed to order 4 and assumes the form

$$
\hat{M}=
M_0+\frac{\Theta^2}{R^2}\sum_{1\le n\le 4}\frac{\epsilon^n}{n!}\left(\frac{\alpha}{p}\right)^2\sum_{i\le j\le n/2}e^{2j}\sum_{1\le k\le j}M_{n,j,k}(s^2,\cos 2kg)\qquad (2)
$$

This Hamiltonian, which we designate the parallax Hamiltonian, was first developed by the second author in 1978 using the Amdahl computer at the University of Cincinnati. It was later reproduced by the first author on the TI-ASC computer at the Naval Research Laboratory.

Extracting the terms from $\hat{M}$ which are independent of the argument of perigee we get the intermediary Hamiltonian

$$
R_2=M_0+\frac{\Theta^2}{r^2}\sum_{1\le n\le 4}\frac{\epsilon^n}{n!}\left(\frac{\alpha}{p}\right)^{2n}\sum_{1\le j\le n/2}e^{2j}M_{n,j,0}(s^2)
$$

The normalization of R_2, which is called the radial intermediary, was described in Ref. 12. The present paper represents an extension of this normalization procedure to the full parallax Hamiltonian $\hat{M}$.

To normalize $\hat{M}$ a second canonical transformation is constructed to transform from the Whittaker variables $(r,\theta,\nu,R,\Theta,N)$ to a set of intermediate Delaunay elements (l',g',h',L',G',H') in such a way that the transformed Hamiltonian depends only on the actions (L',G',H') and the argument of perigee g'. In the following text we will not maintain any special notation such as primed variables to represent the different sets of averaged elements. In general, Delaunay's variables (l,g,h,L,G,H) will be used throughout and the context in which they are used will make it clear which set of averaged elements they represent.

We find from Deprit[13] that the entries in the Lie triangle are given recursively by

$$
J_{i,j}=J_{i+1,j-1}+\sum_{k=0}^{i}\binom{i}{k}(J_{k,j-1};\ W_{i-k+1})\qquad (3)
$$

The first column $(J_{n,0})$ of the triangle is, of course, initialized by the function to be transformed, which in our case is given by Eq. (2). If we assume that the generator

$$
W=\sum_{i\ge 0}\left(\frac{\epsilon^i}{i!}\right)W_{i+1}
$$

and the rows of the triangle are known through order $n-1$, then all the Poisson brackets in Eq. (3) can be evaluated except for $(J_{0,0};\ W_n)$. Thus for $i=0, j=n$, Eq. (3) defines the partial differential equation

$$
J_{0,n}=(J_{0,0};\ W_n)+\tilde{J}_{0,n}\qquad (4)
$$

where $\tilde{J}_{0,n}$ consists of the known quantities from Eq. (3). The function $J_{0,n}$ is chosen to be the average

$$
J_{0,n}=\frac{1}{2\pi}\int_0^{2\pi}\tilde{J}_{0,n}\mathrm{d}l
$$

Then W_n is found by solving Eq. (4). In our case this amounts to forming the quadrature

$$
W_n=\left[\frac{\partial J_{0,0}}{\partial L}\right]^{-1}\int(\tilde{J}_{0,n}-J_{0,n})\mathrm{d}l
$$

Of course this determines W_n only up to an additive function independent of the mean anomaly. Contrary to what Kozai did, we elected not to add terms in sin $2g$ and sin $4g$ to the generator.

To illustrate the procedure for order 1 we have

$$
J_{0,0}=M_0=-\mu^2/2L^2,\qquad J_{1,0}=(\Theta^2/r^2)(\alpha/p)^2(\tfrac{1}{2}-\tfrac{3}{4}s^2)
$$

From the equations of a Delaunay transformation we have the following:

$$
G=\Theta,\quad \frac{\mathrm{d}f}{\mathrm{d}l}=\frac{p^2}{r^2\eta^3},\quad \frac{\partial J_{0,0}}{\partial L}=\frac{\mu^2}{L^3}=\frac{\eta^3 G}{p^2},\quad \eta=\sqrt{1-e^2}
$$

Then

$$
J_{0,1}=\frac{1}{2\pi}\int_0^{2\pi}\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^2\left(\frac{1}{2}-\frac{3}{4}s^2\right)\mathrm{d}l=\frac{G^2\eta^3}{p^2}\left(\frac{\alpha}{p}\right)^2\left(\frac{1}{2}-\frac{3}{4}s^2\right)
$$

and the generator of first order becomes

$$
\begin{aligned}
W_1&=\frac{p^2}{\eta^3 G}\int\left(\frac{\alpha}{p}\right)^2\left(\frac{1}{2}-\frac{3}{4}s^2\right)\left(\frac{G^2\eta^3}{p^2}-\frac{\Theta^2}{r^2}\right)\mathrm{d}l\\
&=G\phi\left(\frac{1}{2}-\frac{3}{4}s^2\right)
\end{aligned}
$$

Here we define $\phi=f-1$ to be the equation of the center.

Through order 2 it was possible to complete this procedure by hand, but beyond that point processing by computer was required. At each order it was impossible to anticipate the elements appearing in the provisional term $\tilde{J}_{0,n}$. Therefore the program stopped execution each time it had constructed and printed $\tilde{J}_{0,n}$. We then analyzed how to construct W_n, and this information was coded into the program. The program was then restarted to proceed to the next order. This procedure required many trials at each order. The elimination of one set of terms from $\tilde{J}_{0,n}$ would often give rise to others that had to be added back into $\tilde{J}_{0,n}$ to be eliminated in a subsequent iteration. We would have welcomed at this point the possibility of entering into an interactive mode with the program to introduce various terms in the generator and receive immediate feedback on how these terms affected the normalization. As yet we know of no such system sufficiently large and flexible enough to handle a problem of this magnitude.

Several sets of elements were encountered, through order 4, that required individual consideration in the generator.

Groups of terms of the form

$$
(\alpha/p)^{2n}(\Theta^2/r^2)F(e,s^2)\cos(kg+\beta f)
$$

where F is a power series in e and s^2, were the easiest to handle. For these groups of elements the identity

$$[M_0;(G/\beta)\sin(kg+\beta f)] = -(\Theta/r)^2\cos(kg+\beta f) \quad (5)$$

produced the term $(G/\beta)F(e,s^2)\sin(kg+\beta f)$ for the generator with no contribution to the normalized Hamiltonian.

Terms such as

$$\phi^{n-1}\frac{\Theta^2}{r^2}\left[\begin{array}{c}\cos\\ \sin\end{array}\right]kg \quad (6)$$

were processed by applying the identity

$$\left(M_0;\frac{\phi^n}{k}G\left[\begin{array}{c}\cos\\ \sin\end{array}\right]kg\right) = -\phi^{n-1}\left(\frac{\Theta^2}{r^2}-\frac{G^2\eta^3}{p^2}\right)\left[\begin{array}{c}\cos\\ \sin\end{array}\right]kg \quad (7)$$

The only contribution to the Hamiltonian occurred for $n=1$. For $n>1$ a term such as

$$\phi^{n-1}\left(\frac{\Theta^2}{r^2}\right)F(e,s^2)\left[\begin{array}{c}\cos\\ \sin\end{array}\right]kg$$

was always found to be paired with the term

$$-\phi^{n-1}\left(\frac{G^2\eta^3}{p^2}\right)F(e,s^2)\left[\begin{array}{c}\cos\\ \sin\end{array}\right]kg$$

As a result of cancellations the only terms like those in Eq. (6) that survived through order 2 were for $n=1$ and $k=0$. But at order 3 we found such terms for $n=2$ and $k=2$, and again at order 4 for $n=2,3$ and $k=2,4$. It was precisely the integration of ϕ terms like those in Eq. (6) that prevented previous efforts at extending Brouwer's theory beyond order 2. It was a consequence of the simplifications introduced by the elimination of the parallax transformation that these terms could now be integrated in closed form.

The two terms

$$\left(\frac{\alpha}{p}\right)^{2n}\left(\frac{G^2\eta^3}{p^2}\right)\cos(kg+\beta f) \text{ and } \left(\frac{\alpha}{p}\right)^{2n}\beta\phi\left(\frac{\Theta^2}{r^2}\right)\sin(kg+\beta f)$$

were found to always occur in tandem. They were normalized through the application of the identity

$$\left[M_0;\phi G\cos(kg+\beta f)+\left(\frac{G}{\beta}\right)\sin(kg+\beta f)\right] = \frac{G^2\eta^3}{p^2}\cos(kg+\beta f)+\beta\phi\frac{\Theta^2}{r^2}\sin(kg+\beta f)$$

A particularly troublesome set of terms encountered was of the form

$$\left(\frac{\alpha}{p}\right)^{2n}\frac{G^2\eta^3}{p^2}\left[\frac{e^2}{4}\cos(2g+2f)+e\cos(2g+f) + \frac{e^2}{4}\cos(2g-2f)+e\cos(2g-f)\right]$$

Elimination of these terms was accomplished by the identity

$$(M_0;\phi G\cos 2g) = -\left(\frac{\Theta^2}{r^2}-\frac{G^2\eta^3}{p^2}\right)\cos 2g$$
$$= -\left(\frac{G}{p}\right)^2\{(1+e\cos f)^2-\eta^3\}\cos 2g$$
$$= -\left(\frac{G}{p}\right)^2\left\{\left(\frac{1}{2}+\frac{\eta^2}{2}-\eta^3\right)\cos 2g+\frac{e^2}{4}\cos(2g+2f) \ldots\right.$$
$$\left.\ldots + e\cos(2g+f)+\frac{e^2}{4}\cos(2g-2f)+e\cos(2g-f)\right\}$$

There were various other groups of terms in the provisional elements $\tilde{J}_{0,n}$ which were systematically eliminated through the application of identities similar to those described here.

Due to the different normalization procedures and because the short period generators are determined only up to the addition of an arbitrary function of g, our long period Hamiltonian to order 3 was not precisely the same as that derived by Kozai. However, our long period Hamiltonians are equivalent; we checked that adding

$$G\left(\frac{\alpha}{p}\right)^4\sin 2g\left\{\left(-\frac{45}{64}\eta^3+\frac{1053}{512}\eta^2+\frac{21}{64}\eta-\frac{861}{512}\right)s^4 + \left(\frac{21}{32}\eta^3-\frac{447}{256}\eta^2-\frac{13}{32}\eta+\frac{383}{256}\right)s^2\right\}$$

to the W_2 of our second short period transformation reproduces Kozai's Hamiltonian.

During construction of the entries for the Lie triangle, it was found that a premature expansion of the factor Θ^2/r^2 into the equivalent

$$(G^2/p^2)(1-e^2/2+2e\cos f+e^2/2\cos 2f)$$

in terms of Delaunay's elements, would have severely complicated the solution of the partial differential, Eq. (4). For this reason the program maintained automatically the entries $(J_{i,j})$ in the triangle as the sum $J_{i,j}=J^{\#}_{i,j}+J^{*}_{i,j}$, where $J^{\#}_{i,j}$ were completely normalized, and $J^{*}_{i,j}$ were unnormalized terms having the general form

$$J^{*}_{i,j} = \frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^{2(i+j)}\sum_{k=0}^{2}\sum_{n\geq -3} f_{n,k}[e,s^2,\cos(2kg+nf)] + \frac{G^2}{p^2}\left(\frac{\alpha}{p}\right)^{2(i+j)}\sum_{m=0}^{2}\phi^m\sum_{k=0}^{2}\sum_{n\geq -3} h_{n,k}[e,s^2,\cos(2kg+nf)] \quad (8)$$

The predominant terms occurring in $J^{*}_{i,j}$ were those in the first series in Eq. (8). Rather than tying ourselves to Delaunay's elements we found it was convenient to construct the Poisson brackets in either of the two sets of coordinates (Delaunay or Whittaker) that were most appropriate. The bracket $(J_{i,j};W_k)=(J^{\#}_{i,j};W_k)+(J^{*}_{i,j};W_k)$, where $(J^{\#}_{i,j};W_k)$ was computed in Delaunay's elements and $(J^{*}_{i,j};W_k)$ was evaluated in Whittaker's variables. This is justified since a Poisson bracket is invariant with respect to two coordinate systems provided they are related by a canonical transformation. Since the generator was derived only in Delaunay's elements we had to prepare and transfer to the program the dictionary of partial derivatives of Delaunay's elements with respect to the Whittaker variables. We summarize here the partial derivatives we used. To compute the Poisson brackets strictly in terms of Delaunay's elements we used the differentials of the fundamental terms

$$G^2=\mu p \qquad \eta=G/L \qquad e=\sqrt{1-\eta^2} \qquad s=\sqrt{1-H^2/G^2}$$

given by

$$dp=2p\frac{dG}{G} \qquad d\eta=\eta\left(\frac{dG}{G}-\frac{dL}{L}\right) \qquad de=\frac{\eta^2}{e}\left(\frac{dL}{L}-\frac{dG}{G}\right)$$

$$ds=\left(\frac{1}{s}-s\right)\frac{dG}{G}-\sqrt{1-s^2}\frac{dH}{sG}$$

The differential of the true anomaly is

$$df = \frac{p^2}{r^2\eta^3} dl + \frac{\sin f}{\eta^2}(2 + e\cos f) de$$

The differentials of Delaunay's elements with respect to Whittaker's variables are

$$de = -\frac{p}{r^2}\cos f dr + \frac{p}{\Theta}\sin f dR + [(2+e\cos f)\cos f + e]\frac{d\Theta}{\Theta}$$

$$df = \frac{\sin f}{e}\left[\frac{p}{r^2}dr + e\cos f\frac{dR}{R} - (2+e\cos f)\frac{d\Theta}{\Theta}\right]$$

$$d\phi = e\sin f\left[\left(\eta + \beta\frac{p^2}{r^2}\right)\frac{dr}{p} + \frac{r}{p}\left(2\eta + \frac{\beta p}{r}e\cos f\right)\frac{dR}{R} - \beta(2+e\cos f)\frac{d\Theta}{\Theta}\right]$$

where $\beta = 1/(1+\eta)$.

As some measure of the computing effort required to achieve order 4, we observed that the program run on the TI-ASC computer required 712 s of central processor time to complete order 4 for the second short period transformation, as compared to only 13 s for order 3.

We list in Table 1 the number of terms in the Hamiltonians and generators we produced and compare them to the same numbers for Brouwer's theory as extended by Kozai.

The long period Hamiltonian through order 2 is

$$\text{LP} = -\frac{\mu^2}{2L^2} + \epsilon\left(\frac{G^2}{p^2}\right)\left(\frac{\alpha}{p}\right)^2\eta^3\left(\frac{1}{2} - \frac{3}{4}s^2\right)$$
$$+\left(\frac{\epsilon^2}{2}\right)\left(\frac{G^2}{p^2}\right)\left(\frac{\alpha}{p}\right)^4\left\{-\frac{15}{8}\eta^3 - \frac{3}{4}\eta^4 + \frac{3}{8}\eta^5 + \frac{15}{4}s^2\eta^3\right.$$
$$+\frac{9}{4}s^2\eta^4 - \frac{3}{8}s^2\eta^5 - \frac{105}{64}s^4\eta^3 - \frac{27}{16}s^4\eta^4 - \frac{15}{64}s^4\eta^5$$
$$\left.+\left(\frac{21}{16}s^2\eta^3 - \frac{21}{16}s^2\eta^5 - \frac{45}{32}s^4\eta^3 + \frac{45}{32}s^4\eta^5\right)\cos 2g\right\}$$

The corresponding generator through order 1 is

$$W = G\left(\frac{\alpha}{p}\right)^2\phi\left(\frac{1}{2} - \frac{3}{4}s^2\right)$$
$$+\epsilon G\left(\frac{\alpha}{p}\right)^4\left\{\phi\left[-\frac{15}{8} + \frac{3}{8}\eta^2 + \frac{15}{4}s^2 - \frac{3}{8}s^2\eta^2 - \frac{105}{64}s^4\right.\right.$$
$$\left.-\frac{15}{64}s^4\eta^2 + \left(\frac{21}{16}s^2 - \frac{21}{16}s^2\eta^2 - \frac{45}{32}s^4 + \frac{45}{32}s^4\eta^2\right)\cos 2g\right]$$
$$\left.+e\beta\left(-\frac{1}{2} + \frac{3}{2}s^2 - \frac{9}{8}s^4\right)\sin f + e^2\beta\left(-\frac{1}{8} + \frac{3}{8}s^2 - \frac{9}{32}s^4\right)\sin 2f\right\}$$

III. Long Period Elimination

After elimination of the short period terms, the long period Hamiltonian is expressed in Delaunay's elements. The general form of the Hamiltonian is

$$\text{LP} = -\frac{\mu^2}{2L^2} + \epsilon\frac{G^2}{p^2}\left(\frac{\alpha}{p}\right)^2\eta^3\left(\frac{1}{2} - \frac{3}{4}s^2\right)$$
$$+\frac{G^2}{p^2}\sum_{n=2}^{3}\frac{\epsilon^n}{n!}\left(\frac{\alpha}{p}\right)^{2n}\sum_{k=0}^{1}\sum_{i=3}^{6}\eta^i f^n_{k,i}(s^2)\cos 2kg$$
$$+\frac{\epsilon^4}{4!}\frac{G^2}{p^2}\left(\frac{\alpha}{p}\right)^8\sum_{k=0}^{2}\sum_{i=3}^{8}f^3_{k,i}(s^2)\cos 2kg \tag{9}$$

where the functions $f^2_{k,i}$, $f^3_{k,i}$, and $f^4_{k,i}$ are polynomials in s^2.

The elimination of the long period terms from LP was accomplished by yet another canonical transformation of the Lie type. This transformation was far simpler to construct than for the short period terms since Eq. (9) defines an integrable system with but one degree of freedom.

Since LP is secular through order 1, the Lie Eq. (4) dictates that we impose the condition $\partial W_1/\partial l = 0$. This guarantees that the final secular Hamiltonian will agree with LP through order 1. In general, in order not to reintroduce short period terms we impose the condition that the generator at all orders be independent of the mean anomaly.

At order 2 the differential Eq. (4) becomes

$$J_{0,2} = J_{2,0} + 2(J_{1,0};\ W_1) + (J_{0,0};\ W_2)$$
$$= J_{2,0} + 2(J_{1,0};\ W_1)$$

The term $J_{0,2}$ is formed by choosing the secular terms from $J_{2,0}$. W_1 is determined by the quadrature

$$W_1 = \frac{1}{2}\Big/\left(\frac{\partial J_{1,0}}{\partial G}\right)\int (J_{2,0} - J_{0,2})dg$$

Of course we now encounter the critical inclination since

$$\frac{\partial J_{1,0}}{\partial G} = \frac{3}{4}\frac{G}{p^2}\left(\frac{\alpha}{p}\right)^2 n^3(1 - 5c^2)$$

The remaining entries in the triangle are filled according to Eq. (3) and each order for the generator and the secular Hamiltonian is derived as we have described here for the first order.

As expected, the critical inclination does not appear as a small divisor in the Hamiltonian until order 3, but is present in every order of the generator. The secular Hamiltonian we produced agrees through order 3 with Kozai's.

Table 1 Number of terms

	Elimination of the parallax		Second short period transformation		Brouwer-Kozai von Zeipel	
Order	Hamiltonian	Generator	Hamiltonian	Generator	Hamiltonian	Generator
1	2	7	2	2	2	7
2	8	32	13	16	15	213
3	11	112	28	125	28	2076[a]
4	26	264	72	612	?	?

[a] Estimated from the number of terms averaged over the mean anomaly.

Table 2 Fourth-order secular Hamiltonian

Coefficient	$1-5c^2$	Sin I	η	Coefficient	$1-5c^2$	Sin I	η	Coefficient	$1-5c^2$	Sin I	η	Coefficient	$1-5c^2$	Sin I	η
-89775 / 4096	0	8	8	4683825 / 1024	-2	10	4	-402975 / 256	-1	8	8	3898125 / 4096	-3	14	3
-2324295 /16384	0	8	7	23929425 / 4096	-2	10	3	-582795 / 2048	-1	8	7	-941625 / 4096	-3	12	9
943245 / 2048	0	8	6	54675 / 512	-2	8	9	44145 / 16	-1	8	6	-1974375 / 512	-3	12	8
-2223495 / 8192	0	8	5	-1534815 / 256	-2	8	8	1379025 / 1024	-1	8	5	-17998875 / 4096	-3	12	7
-2112075 / 4096	0	8	4	-16065855 / 2048	-2	8	7	-287955 / 256	-1	8	4	1974375 / 256	-3	12	6
-2346795 /16384	0	8	3	1558575 / 128	-2	8	6	-2298375 / 2048	-1	8	3	38822625 / 4096	-3	12	5
17955 / 512	0	6	8	17465355 / 1024	-2	8	5	394875 / 256	-1	6	8	-1974375 / 512	-3	12	4
286335 / 1024	0	6	7	-1582335 / 256	-2	8	4	110493 / 256	-1	6	7	-19882125 / 4096	-3	12	3
-143055 / 128	0	6	6	-19083555 / 2048	-2	8	3	-85401 / 32	-1	6	6	-91125 / 1024	-3	10	9
84393 / 512	0	6	5	-81459 / 256	-2	6	9	-209223 / 128	-1	6	5	1703025 / 256	-3	10	8
745389 / 512	0	6	4	970605 / 256	-2	6	8	275481 / 256	-1	6	4	10204425 / 1024	-3	10	7
1308375 / 1024	0	6	3	3297069 / 512	-2	6	7	320805 / 256	-1	6	3	-1703025 / 128	-3	10	6
-8883 / 512	0	4	8	-981441 / 128	-2	6	6	-19215 / 32	-1	4	8	-20135475 / 1024	-3	10	5
-70011 / 512	0	4	7	-1658079 / 128	-2	6	5	-14643 / 64	-1	4	7	1703025 / 256	-3	10	4
255825 / 256	0	4	6	992277 / 256	-2	6	4	8127 / 8	-1	4	6	10022175 / 1024	-3	10	3
57735 / 128	0	4	5	3498165 / 512	-2	6	3	47997 / 64	-1	4	5	556605 / 1024	-3	8	9
-789399 / 512	0	4	4	88641 / 512	-2	4	9	-3213 / 8	-1	4	4	-730755 / 128	-3	8	8
-1183437 / 512	0	4	3	-134127 / 128	-2	4	8	-8559 / 16	-1	4	3	-11114685 / 1024	-3	8	7
189 / 32	0	2	8	-1165815 / 512	-2	4	7	2205 / 32	-1	2	8	730755 / 64	-3	8	6
-2727 / 256	0	2	7	135009 / 64	-2	4	6	63 / 2	-1	2	7	20559555 / 1024	-3	8	5
-13725 / 32	0	2	6	2185659 / 512	-2	4	5	-441 / 4	-1	2	6	-730755 / 128	-3	8	4
-63621 / 128	0	2	5	-135891 / 128	-2	4	4	-1449 / 16	-1	2	5	-10001475 / 1024	-3	8	3
12123 / 16	0	2	4	-1108485 / 512	-2	4	3	1323 / 32	-1	2	4	-116865 / 256	-3	6	9
400041 / 256	0	2	3	-3087 / 128	-2	2	9	945 / 16	-1	2	3	155925 / 64	-3	6	8
-189 / 64	0	0	8	1323 / 16	-2	2	8	-455625 / 4096	-2	12	9	1470105 / 256	-3	6	7
1215 / 128	0	0	7	30429 / 128	-2	2	7	-2521125 / 2048	-2	12	8	-155925 / 32	-3	6	6
2655 / 32	0	0	6	-1323 / 8	-2	2	6	-6773625 / 8192	-2	12	7	-2589615 / 256	-3	6	5
8235 / 64	0	0	5	-51597 / 128	-2	2	5	2581875 / 1024	-2	12	6	155925 / 64	-3	6	4
-9585 / 64	0	0	4	1323 / 16	-2	2	4	4748625 / 2048	-2	12	5	1236375 / 256	-3	6	3
-45045 / 128	0	0	3	24255 / 128	-2	2	3	-2642625 / 2048	-2	12	4	15435 / 128	-3	4	9
577125 / 1024	-1	10	8	455625 / 4096	-3	14	9	-11309625 / 8192	-2	12	3	-6615 / 16	-3	4	8
50625 / 1024	-1	10	7	455625 / 512	-3	14	8	710775 / 4096	-2	10	9	-152145 / 128	-3	4	7
-127575 / 128	-1	10	6	2986875 / 4096	-3	14	7	4505625 / 1024	-2	10	8	6615 / 8	-3	4	6
-378675 / 1024	-1	10	5	-455625 / 256	-3	14	6	17496675 / 4096	-2	10	7	257985 / 128	-3	4	5
419175 / 1024	-1	10	4	-7340625 / 4096	-3	14	5	-4594725 / 512	-2	10	6	-6615 / 16	-3	4	4
176175 / 512	-1	10	3	455625 / 512	-3	14	4	-42136875 / 4096	-2	10	5	-121275 / 128	-3	4	3

The generator corresponding to the secular Hamiltonian is given to order 1 by

$$W=G\left(\frac{\alpha}{p}\right)^2\left(\frac{1}{1-5c^2}\right)\left(\frac{7}{16}s^2-\frac{7}{16}s^2\eta^2-\frac{15}{32}s^4+\frac{15}{32}s^4\eta^2\right)\sin 2g+\epsilon G\left(\frac{\alpha}{p}\right)^4\left\{\left(\frac{1}{1-5c^2}\right)^2\left(-\frac{385}{32}s^2-\frac{21}{4}s^2\eta+\frac{217}{16}s^2\eta^2+\frac{21}{4}s^2\eta^3\right.\right.$$

$$-\frac{49}{32}s^2\eta^4+\frac{1165}{32}s^4+\frac{321}{16}s^4\eta-\frac{1249}{32}s^4\eta^2-\frac{321}{16}s^4\eta^3+\frac{21}{8}s^4\eta^4-\frac{9145}{256}s^6-\frac{405}{16}s^6\eta+\frac{2275}{64}s^6\eta^2+\frac{405}{16}s^6\eta^3+\frac{45}{256}s^6\eta^4+\frac{5775}{512}s^8$$

$$\left.+\frac{675}{64}s^8\eta-\frac{1275}{128}s^8\eta^2-\frac{675}{64}s^8\eta^3-\frac{675}{512}s^8\eta^4\right)\sin 2g+\left(\frac{1}{1-5c^2}\right)\left(-\frac{15}{8}s^2-\frac{21}{16}s^2\eta+s^2\eta^2+\frac{35}{16}s^2\eta^3+\frac{59}{16}s^4+\frac{27}{8}s^4\eta\right.$$

$$\left.-\frac{23}{16}s^4\eta^2-\frac{45}{8}s^4\eta^3-\frac{435}{256}s^6-\frac{135}{64}s^6\eta+\frac{75}{256}s^6\eta^2+\frac{225}{64}s^6\eta^3\right)\sin 2g+\left(\frac{1}{1-5c^2}\right)^3\left(-\frac{245}{256}s^4+\frac{245}{128}s^4\eta^2-\frac{245}{256}s^4\eta^4+\frac{385}{128}s^6-\frac{385}{64}s^6\eta\right.$$

$$\left.+\frac{385}{128}s^6\eta^4-\frac{3225}{1024}s^8+\frac{3225}{512}s^8\eta^2-\frac{3225}{1024}s^8\eta^4+\frac{1125}{1024}s^{10}-\frac{1125}{512}s^{10}\eta^2+\frac{1125}{1024}s^{10}\eta^4\right)\sin 4g$$

$$\left.+\left(\frac{1}{1-5c^2}\right)^2\left(\frac{49}{128}s^4-\frac{49}{64}s^4\eta^2+\frac{49}{128}s^4\eta^4-\frac{105}{128}s^6+\frac{105}{64}s^6\eta^2-\frac{105}{128}s^6\eta^4+\frac{225}{512}s^8-\frac{225}{256}s^8\eta^2+\frac{225}{512}s^8\eta^4\right)\sin 4g\right\}$$

We provide in Table 2 the coefficients for the terms in the series

$$\frac{G^2}{p^2}\left(\frac{\alpha}{p}\right)^8\sum_{i=1}^{144}a_i(1-5c^2)^{i_1}\eta^{i_2}$$

representing the new fourth-order secular Hamiltonian. The coefficients a_i are given by the fractions in the left column followed by the exponents for the quantities that head the columns. This table was composed and printed by computer directly from the output of the program that produced the results; thus it is free of the usual errors associated with transcribing results of this nature.

IV. Conclusion

A closed-form third-order solution to the main problem in the theory of an artificial satellite is developed. The theory begins with the preparation of an intermediate Hamiltonian by the elimination of the parallax transformation. The reduction by this transformation of the factors $(1/r)^n$ to $(1/r)^2$ and the concentration of the short period terms in the factor $(1/r)^2$ accrues enormous simplifications throughout the normalization procedure. Following this transformation we construct a second transformation to eliminate the remaining short period terms.

A comparison with Brouwer's theory, where the short period terms are eliminated via a single transformation, points to the value of the parallax transformation. In Brouwer's theory one experiences a rapid buildup in the unnormalized terms stemming principally from the $(1/r)^3$ term at order 1 and culminating in an enormous number of terms at order 3, which effectively limits the short period elimination at that order. On the other hand the elimination of the parallax transformation holds in check this buildup of terms and allows our short period elimination to proceed to order 4. An additional aspect of the parallax transformation that assists the normalization is that the quantity $(1/r)^2$ is a natural expression to have in the Hamiltonian since r^2 is proportional to $\partial f/\partial l$, and the generator of the second transformation is developed as a function of true anomaly f.

In the second transformation we found that construction of the generator was much facilitated by encouraging the expressions coming from the Poisson brackets to have the factor $(1/r)^2$. The factor $(1/r)^2$ was generated automatically by the Poisson brackets involving the parallax Hamiltonian, since in this case the brackets were computed in Whittaker's variables. This is in contrast to the Poisson brackets involving the transformed Hamiltonian which were more naturally computed in Delaunay's elements. Actually at times it was necessary to combine terms arising from various brackets into the factor $(1/r)^2$ to aid in recognizing the appropriate expressions for the generator. The calculation of the Poisson brackets in two equivalent sets of canonical variables is a radical departure from past practice. Indeed past practice would lead one to believe that a normalization must be conducted in one set of canonical variables to the exclusion of any other. We have on the one hand Brouwer's theory exclusively developed in Delaunay's elements, whereas Aksnes[14] proposed to use exclusively the so-called modified Hill variables. The truth of the matter is that one should exploit the freedom guaranteed by the invariance of Poisson brackets with respect to canonical variables. In our case the elimination of the parallax transformation is computed completely within the space of Whittaker's variables. This is in contrast with the already mentioned mixed computations followed in the second transformation.

References

[1]Deprit, A. and Rom, A., "The Main Problem of Artificial Satellite Theory for Small and Moderate Eccentricities," *Celestial Mechanics*, Vol. 2, 1970, pp. 166-206.

[2]Claes, H., "Analytical Theory of Earth's Artificial Satellites," *Celestial Mechanics*, Vol. 21, 1980, pp. 193-198.

[3]Kinoshita, H. "Third-Order Solution of an Artificial Satellite Theory," Smithsonian Astrophysical Observatory, Rept. 379, 1977.

[4]Brouwer, D., "Solution of the Problem of Artificial Satellite Theory Without Drag," *Astronomical Journal*, Vol. 64, 1959, pp. 378-397.

[5]Kozai, Y., "Second-Order Solution of Artificial Satellite Theory Without Air Drag," *Astronomical Journal*, Vol. 67, 1962, pp. 446-461.

[6]Hoots, F.R., "A Nonsingular Reformulation of the Brouwer Geopotential Theory," Paper 79-137, AAS/AIAA Astrodynamics Conference, Provincetown, Mass., June 25-27, 1979.

[7]Aksnes, K., "A Second-Order Artificial Satellite Theory Based on an Intermediate Orbit," *Astronomical Journal*, Vol. 75, 1970, pp. 1066-1076.

[8]Jefferys, W.H., "Automated, Closed Form Integration of Formulas in Elliptic Motion," *Celestial Mechanics*, Vol. 3, 1971, pp. 390-394.

[9]Rom, A., "Mechanized Algebraic Operations (MAO)," *Celestial Mechanics*, Vol. 1, 1970, pp. 301-319.

[10]Dasenbrock, R.R., "Algebraic Manipulation by Computer," Naval Research Laboratory, Washington D.C., Rept. 73-1, June 29, 1973.

[11]Deprit, A., "The Elimination of the Parallax in Satellite Theory," *Celestial Mechanics*, Vol. 24, 1981, pp. 111-153.

[12]Deprit, A., "The Main Problem in the Theory of Artificial Satellites to Order Four," *Journal of Guidance and Control*, Vol. 4, March-April 1981, pp. 201-206.

[13]Deprit, A., "Canonical Transformations Depending on a Small Parameter," *Celestial Mechanics*, Vol. 1, 1969, pp. 12-30.

[14]Aksnes, K., "On the Use of the Hill Variables in Artificial Satellite Theory: Brouwer's Theory," *Astronomy and Astrophics*, Vol. 17, 1975, pp. 70-75.